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Abstract. This paper establishes the Rosseland approximation of the radiative transfer equations in a gray atmosphere — i.e. assuming that the opacity is independent of the radiation frequency. The problem is set in a smooth bounded domain Ω of the Euclidian space $\mathbf{R}^3$, assuming that the incoming radiation intensity at the boundary of Ω is Lipschitz continuous and isotropic — including for instance the case of a black-body radiation with nontrivial temperature gradient.

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1. Presentation of the model

Radiative transfer is an important phenomenon in many branches of physics and, in many situations, the most significant mode of transmission of energy.

As established in all textbooks on radiation hydrodynamics (see, for example, [15]), radiative transfer in a host medium can be described by the following set of partial differential equations, with unknowns the *radiative intensity* $I = I(t, x, \omega, \nu)$ and the *temperature* $T = T(t, x)$. The independent variables in these equations are the time $t \in [0, \tau]$, $\tau > 0$, the position $x \in \Omega \subseteq \mathbf{R}^3$, the spherical angle with respect to a suitable reference frame $\omega \in S^2$ and the frequency $\nu \in \mathbf{R}^+$. With these notations, the radiative transfer equations are

$$
(1) \qquad \begin{cases} \dfrac{1}{c}\dfrac{\partial I}{\partial t} + \omega \cdot \nabla_x I + \Sigma(T, \nu)(I - B_\nu(T)) = 0 \\[2ex] \dfrac{\partial E}{\partial t}(T) = \displaystyle\int_0^{+\infty} \Sigma(T, \nu)(\bar{I} - B_\nu(T))\, d\nu, \end{cases}
$$

where

$$
B_\nu = \frac{\alpha \nu^3}{e^{h\nu/kT} - 1} \qquad \bar{I} = \frac{1}{4\pi}\int_{S^2} I(t, x, \omega, \nu)\, d\omega.
$$

The parameter $c > 0$ is the speed of light and $B_\nu(T)$ is Planck's formula for the black-body radiation at temperature T, the parameters h and k being the Planck and the Boltzmann constants respectively.

We have introduced the normalization constant

$$\alpha = 15 \left(\frac{h}{k\pi}\right)^4 ,$$

so that Stefan's law assumes the simple form:

$$\int_0^{+\infty} B_\nu(T)\, d\nu = T^4.$$

The function $\Sigma = \Sigma(T, \nu)$, is the *opacity* — or absorption cross-section — of the host medium. In general, the opacity takes into account how energy is transferred from the incoming radiation to the electrons in the background medium. Its expression is complicated in general, as it must take into account the contributions of bound electrons, of free electrons as well as transitions from bound to free states. The respective importance of each such transition in the opacity strongly depends upon the nature of the background and the temperature. (For instance, bound-bound transitions are more important for heavy elements; however, their importance decreases at high temperatures as most electrons are in free states). We refer to chapter VII of [18] and to [1] for more information on this subject.

Thus we seek to establish the validity of the Rosseland approximation under the mildest possible assumptions on the opacity, since so little is known about it in general.

There are however, two particular features of the opacity that complicate the Rosseland approximation, and that we wish to take into account in the present paper.

Specifically, our proof of the Rosseland approximation allows considering opacities that may tend to infinity as $T \to 0^+$. As we shall see, such opacities lead to a limit equation that is parabolic degenerate, and hence has singular solutions.

This kind of behavior of the opacity can be seen on the example of the Kramers opacity (an explicit formula for the opacity in the case of free-free transitions and for a hydrogen-like atom). The Kramers opacity is of the form

$$\Sigma_K(T, \nu) = C \frac{1 - e^{-\frac{h\nu}{kT}}}{(h\nu)^3 (kT)^{1/2}} \left(1 + O\left(\frac{kT}{h\nu}\right)\right)$$

— see fla (7.72) on p. 174 in [18]. In this example $\Sigma(T, \nu) \sim T^{-1/2}$.

Otherwise, the opacity can oscillate strongly because of bound-free and bound-bound transitions.

In view of the above considerations, we only assume in this paper that the opacity satisfies $\Sigma(T, \nu) \sim T^{-\lambda}$ as $T \to 0$, with $\lambda \in (0, 1]$.

The other important nonlinear term in system (1) is the *internal energy* $E = E(T)$. This quantity depends on the temperature, but the experimental curves suggest that its dependence upon T is simpler than that of the cross section.

Henceforth, we assume that E is increasing and such that $E'(T) \sim T^\gamma$ as $T \to 0$, where $\gamma \in [0, 2]$. This assumption includes the perfect gas law, $E(T) = c_V T$. Without loss of generality we further assume that $E(0) = 0$.

In some relevant physical situations, for example in the external layers of a stellar atmosphere [15], the opacity is independent of the photons frequency. It is therefore convenient to integrate system (1) over frequencies and obtain the following system of equations, called the *gray model*:

$$
\begin{cases} \dfrac{1}{c}\dfrac{\partial u}{\partial t} + \omega \cdot \nabla_x u + \Sigma(T)(u - T^4) = 0 \\ \dfrac{\partial E}{\partial t}(T) = \Sigma(T)(\bar{u} - T^4), \qquad \bar{u} = \dfrac{1}{4\pi}\displaystyle\int_{S^2} u(t, x, \omega)\, d\omega. \end{cases} \tag{2}
$$

Here, the unknowns are the *photon density*

$$
u = u(t, x, \omega) = \int_0^{+\infty} I(t, x, \omega, \nu)\, d\nu
$$

and, again, the temperature $T = T(t, x)$.

In this paper, we restrict our attention to the gray system (2). The mathematical study of system (1) will be left for future research.

We consider situations where the interaction between radiation and matter is the dominant phenomenon.

Through multiple interactions, the photon experiences a sort of random walk in the host medium. Since the equilibrium (Planck's function) has mean zero velocity, it is therefore natural to investigate a diffusive regime for the photons gas.

From a mathematical point of view, this would mean that it is possible to obtain diffusion-type equations through the usual parabolic scaling:

$$
\begin{cases} \dfrac{\varepsilon}{c}\dfrac{\partial u_\varepsilon}{\partial t} + \omega \cdot \nabla_x u_\varepsilon + \dfrac{1}{\varepsilon}\Sigma(T_\varepsilon)(u_\varepsilon - T_\varepsilon^4) = 0 \\ \varepsilon^2 \dfrac{\partial E}{\partial t}(T_\varepsilon) = \Sigma(T_\varepsilon)(\bar{u}_\varepsilon - T_\varepsilon^4), \qquad \bar{u}_\varepsilon = \dfrac{1}{4\pi}\displaystyle\int_{S^2} u_\varepsilon(t, x, \omega) d\omega. \end{cases} \tag{3}
$$

Here $\varepsilon > 0$ is a non-dimensional quantity which represents the ratio between the mean free path of the photons and some characteristic length of the host medium (for example, in the case of the Sun, the mean free path is typically of the order of magnitude of a few centimeters, which is negligeable if compared to the size of the Sun).

It is therefore obvious to look at the behavior of the angular average of the photon density, governed by system (3), when $\varepsilon \to 0$.

The first step in obtaining the formal limit is the definition of the *macroscopic density*

$$\rho_\varepsilon(t,x) = \bar{u}_\varepsilon(t,x,\omega) = \frac{1}{4\pi}\int_{S^2} u_\varepsilon(t,x,\omega)\,d\omega$$

and the *flux* vector $j_\varepsilon(t,x) = (j_\varepsilon^{(1)}(t,x), j_\varepsilon^{(2)}(t,x), j_\varepsilon^{(3)}(t,x))$:

$$j_\varepsilon^{(i)}(t,x) = \frac{1}{4\pi\varepsilon}\int_{S^2} \omega^{(i)} u_\varepsilon(t,x,\omega)\,d\omega,$$

where $\omega^{(i)}$ denotes the i-th component of the vector ω.

Then, by integrating the first equation of System (3), it is easy to deduce that

$$\frac{\partial}{\partial t}\left[\frac{\rho_\varepsilon}{c} + E(T_\varepsilon)\right] + \nabla_x \cdot j_\varepsilon = 0.$$

Moreover, integrating the first equation of System (3) with respect to ω in S^2 after multiplying each side of this equation by $\omega^{(i)}$ gives

$$\frac{\varepsilon^2}{c}\frac{\partial j_\varepsilon^{(i)}}{\partial t} + \frac{1}{4\pi}\int_{S^2} \omega^{(i)}\nabla_x \cdot (\omega u_\varepsilon)\,d\omega + \Sigma(T_\varepsilon)\,j_\varepsilon^{(i)} = 0.$$

Assume the existence of the two limits:

$$\rho = \lim_{\varepsilon\to 0} \rho_\varepsilon, \qquad j = \lim_{\varepsilon\to 0} j_\varepsilon.$$

If j_ε is bounded in some sense and that

$$\lim_{\varepsilon\to 0} \varepsilon^2 E_t(T_\varepsilon) = 0 \qquad \lim_{\varepsilon\to 0} \varepsilon^2 (j_\varepsilon)_t = 0,$$

the second equation of (3) implies that

$$\lim_{\varepsilon\to 0} T_\varepsilon^4 = \rho.$$

At leading order in ε, System (3) forces u to be independent of ω, so that $\rho = u$. Moreover, ρ solves the following diffusion-type equation, which is known in the literature under the name of *Rosseland equation*:

$$\left[\frac{\rho}{c} + E(\rho^{1/4})\right]_t - \nabla_x \cdot \left[\frac{\nabla_x \rho}{3\Sigma(\rho^{1/4})}\right] = 0.$$

One major difficulty with the Rosseland approximation is about boundary conditions. As is well known, natural boundary conditions for transport equations and for diffusion equations are of a very different nature — for instance, Dirichlet boundary conditions are admissible for diffusion equations, not for transport equations. A typical instance of natural boundary condition for a transport equation consists of prescribing the density of particles entering the spatial domain where the equation is to be solved.

So far, the Rosseland approximation has been established in the case of matter radiating in vacuum — see [3].

In the more complicated case of a prescribed incident radiation intensity, the Rosseland approximation is based on asymptotic analysis involving nontrivial boundary layer analysis: see [12], and [2] for a mathematical justification of these asymptotic ansatz, based on somewhat unrealistic monotonicity assumptions on the opacity.

Here, we discuss the case of an incident radiation field that is a Planck distribution (thereby avoiding boundary layers), however with a possible temperature gradient at the boundary. This type of boundary condition cannot be handled with the ideas of either [2, 3] for realistic opacities.

2. Mathematical assumptions and results

We start this section by declaring some notation which we will use in the sequel.

We will work in a bounded domain $\Omega \subset \mathbf{R}^3$ with boundary $\Gamma = \partial\Omega$ of class C^1.

If we denote with n_x the outer normal to the boundary of Γ in x, we can define the sets Γ_-, Γ_+ and Γ_0, subsets of Γ, as:

$$\Gamma_- = \{x \in \Gamma \ : \ \omega \cdot n_x < 0\}, \qquad \Gamma_+ = \{x \in \Gamma \ : \ \omega \cdot n_x > 0\}$$

and

$$\Gamma_0 = \{x \in \Gamma \ : \ \omega \cdot n_x = 0\}.$$

Moreover, since in the paper we will heavily use indices, in order to avoid confusion, any vector $v \in \mathbf{R}^n$ will be written in the following way, when it will be convenient to give the components in explicit form:

$$v = (v^{(1)}, \dots, v^{(n)}).$$

We are now ready to give a brief presentation of our results. Goal of the paper is the mathematical study of system (3): we prove existence of a weak solution and we deduce rigorously the Rosseland approximation as a diffusive limit of such system.

We assume that the opacity and the internal energy satisfy the following properties:

Definition 2.1. *Let $\Sigma = \Sigma(T)$ and $E = E(T)$ be the opacity and the internal energy of system* (3). *We say that Σ and E are admissible if and only if*

(1) $\Sigma(y) > 0$ *for any* $y > 0$ *and is of class* $C^1((0, +\infty))$*;*

(2) $\lim_{y \to +\infty} \Sigma(y) = 0$*;*

(3) $\lim_{y \to 0} \Sigma(y) = +\infty$ *and* $\Sigma(y) \sim y^{-\beta}$ *when* $y \to 0$*, with* $\beta \in (0, 1]$*.*

(4) E *is continuous in* $[0, +\infty)$ *and belongs to the class* $C^1((0, +\infty))$*. Moreover* $E(0) = 0$ *and* $E'(y) \geq 0$ *for all* $y > 0$*.*

(5) $E'(y) \sim y^\alpha$ *for* $y \to 0$*, where* $\alpha \in [0, 2]$*.*

In what follows, we will consider the initial-boundary value problem for System (3), with $(t, x, \omega) \in (0, \tau) \times \Omega \times S^2$, supplemented with appropriate initial and boundary conditions.

Our first theorem concerns an existence result:

Theorem 2.2. *Let us consider System* (3) *for* $(t, x, \omega) \in (0, \tau) \times \Omega \times S^2$, *with initial conditions*

$$u_\varepsilon(0, x, \omega) = u^{\mathrm{in}}(x, \omega) \in L^\infty(\Omega \times S^2)$$

$$T_\varepsilon(0, x) = T^{\mathrm{in}}(x) \in L^\infty(\Omega)$$

and boundary data

$$u_\varepsilon(t, x, \omega)|_{\Gamma_-} = u^{\mathrm{b}}(t, x, \omega) \in L^\infty((0, \tau) \times \partial\Omega \times S^2),$$

where $\Omega \subset \mathbf{R}^3$ *is a bounded domain with boundary of class* C^1 *and* $\varepsilon > 0$. *Then System* (3) *admits a weak solution*

$$(u_\varepsilon(t, x, \omega), T_\varepsilon(t, x)) \in L^\infty((0, \tau) \times \Omega \times S^2) \times L^\infty((0, \tau) \times \Omega).$$

This existence result improves on the earlier references [14, 8] which used unrealistic monotonicity assumptions on the opacity σ, even in the gray case. It also improves on the existence result in [3], which further reduces the system of two equations (2) to a scalar equation on the single unknown u by neglecting the time-derivative of the internal energy in the second equation of (2). An earlier attempt at treating the system (2) can be found in [6], with however a lesser degree of generality in the opacity and internal energy admissible.

Moreover, we give a rigorous justification of the Rosseland limit equation of the radiative transfer system. Here, in order to avoid additional difficulties dues to the presence of a boundary layer, we do not allow that the boundary data depend on ω.

The result is summarized in the following theorem:

Theorem 2.3. *Let us consider System* (3), *posed for* $(t, x, \omega) \in (0, \tau) \times \Omega \times S^2$, *with initial conditions*

$$u_\varepsilon(0, x, \omega) = u^{\mathrm{in}}(x, \omega) \in L^\infty(\Omega \times S^2)$$

$$T_\varepsilon(0, x) = T^{\mathrm{in}}(x) \in L^\infty(\Omega)$$

and boundary data

$$u_\varepsilon(t, x, \omega)|_{\Gamma_-} = u^{\mathrm{b}}(t, x) \in W^{1,\infty}((0, \tau) \times \partial\Omega),$$

where $\Omega \subset \mathbf{R}^3$ *is a bounded domain with boundary of class* C^1 *and* $\varepsilon > 0$. *Let* $\{u_\varepsilon\}$ *be a family of solutions of System* (3) *which satisfy the same initial and boundary conditions.*

Then, there exists a function $\rho \in L^\infty((0,\tau)\times\Omega)$ such that $\rho = \lim_{\varepsilon\to 0}\bar{u}_\varepsilon$ strongly in $L^2((0,\tau)\times\Omega)$. The limit function ρ moreover solves the following initial-boundary value problem in $(0,\tau)\times\Omega$, for $\tau>0$:

$$\left[\frac{\rho}{c} + E(\rho^{1/4})\right]_t - \nabla_x \cdot \left[\frac{\nabla_x \rho}{3\Sigma(\rho^{1/4})}\right] = 0.$$

with initial conditions $\rho(0,x) = \bar{u}^{\mathrm{in}}(x,\omega)$ and boundary data $\rho(t,x)|_{\partial\Omega} = u^{\mathrm{b}}(t,x)$.

For technical reasons, the regularity condition on the boundary data for the family $\{u_\varepsilon\}$ in Theorem 2.3 is slightly more stringent than the requirements on the boundary conditions in Theorem 2.2.

It is interesting to note that, even if only the incoming density on the boundary is prescribed for the family $\{u_\varepsilon\}$, we obtain that the boundary condition for the limit is fully defined.

The Rosseland approximation has been established in [2] by a method based on a multiscale expansion in powers of ε, assuming that the opacity Σ is decreasing, while $T \mapsto \Sigma(T)T^4$ is increasing.

In [3], the same Rosseland approximation for a reduced version of the gray model (2) neglecting the time derivative of $E(T)$ in the energy balance equation. The method in [3] used compactness arguments based on velocity averaging, and hence could handle the most general class of opacities imaginable. However, it assumed an absorbing boundary condition — corresponding to radiation expanding in the vacuum. The same method could also handle the case of a incoming black-body radiation at a constant temperature. The case of a temperature gradient at the boundary, treated in the present paper, requires a new idea for controling the radiation flux, which is the key to the Rosseland approximation.

Finally, we would like to mention the reference [11], where a nonlinear diffusion approximation is established by compensated compactness — at variance with the argument in [3] based on velocity averaging instead.

Because of the conditions on the opacity given in Definition 2.1, the Rosseland equation is degenerate parabolic, so that the temperature may display some singularity in the form of a propagating front when radiation penetrates into cold matter. Hence result above proves that the Rosseland approximation is robust in the sense that it holds even in situations in which such singularities may occur — something that is not at all clear with classical asymptotic expansions.

The method for proving existence of a weak solution (Theorem 2.2) uses a compactness argument based on a maximum principle and velocity averaging. The main difficulty is that the absorption-emission nonlinear term is not a Lipschitz continuous perturbation of the transport operator, since the opacity tends to infinity as the temperature vanishes.

The method for proving the Rosseland approximation is based on using the relative entropy to control the radiation flux. Usually, the relative entropy method is used when solutions of the target equation are known to be smooth. This is not the case here, as temperature may fail to be differentiable at interfaces with cold matter. Here, as in the case of Carleman's kinetic model [10], we use both relative entropy and a compactness argument that allows for possibly singular solutions of the target equations. Relative entropy is not used all the way through the proof to control the distance between the radiative intensity and the (fourth power of) the temperature that solves the Rosseland limiting equation, but only to control the radiation flux. This control is in turn one essential step in the following compactness argument.

That it is possible to use relative entropy within a compactness argument to establish the validity of such a macroscopic (or hydrodynamic) limit seems to be special to the case of limits leading to a pure diffusion equation (degenerate or not), but without streaming term. In particular, the strategy presented here does not seem to apply to the derivation of the incompressible Navier-Stokes equations from the Boltzmann equation.

From a mathematical point of view, it will be more convenient to work with two new unknowns: the photon density $u = u(t, x, \omega)$ and the normalized temperature $\theta = \theta(t, x)$, the last one being defined as $\theta = T^4$. Moreover, we will set $c = 1$.

We write now the radiative transfer equations (3) with respect to the two new unknowns.

We first define the normalized cross section σ and the internal energy $\mathcal{E}$ as

$$\sigma(\theta) = \Sigma(\theta^{1/4}), \qquad \mathcal{E}(\theta) = E(\theta^{1/4}).$$

The pair $(\sigma, \mathcal{E})$, must satisfy the same constraints we have imposed on the pair (Σ, E). The precise meaning of admissible cross section and energy is given in the following definition:

Definition 2.4. *The normalized opacity*

$$\sigma \;:\; \mathbf{R}^+ \to \mathbf{R}^+$$

and the internal energy

$$\mathcal{E} \;:\; [0, +\infty) \to [0, +\infty)$$

are admissible if and only if

(1) $\sigma(y)$ *is strictly positive for any* $y > 0$ *and is of class* $C^1((0, +\infty))$*;*

(2) $\lim_{y \to +\infty} \sigma(y) = 0$*;*

(3) $\lim_{y \to 0} \sigma(y) = +\infty$ *and* $\sigma(y) \sim y^{-\beta}$ *when* $y \to 0$*. We will always suppose that* $\beta \in (0, 1/4]$.

(4) $\mathcal{E}$ *is continuous in* $[0, +\infty)$ *and is of class* $C^1((0, +\infty))$*. Moreover* $\mathcal{E}(0) = 0$ *and* $\mathcal{E}'(y) \geq 0$ *for all* $y > 0$*.*

(5) $\mathcal{E}'(y) \sim y^{\alpha}$ *for* $y \to 0$*, where* $\alpha \in [-3/4, -\beta]$*, where* β *is the same exponent that gives the behavior of* σ *when* $y \to 0$*.*

Then, System (3) becomes, in the new unknowns,

$$\text{(4)} \qquad \begin{cases} \varepsilon^2 \dfrac{\partial u_\varepsilon}{\partial t} + \varepsilon \omega \cdot \nabla_x u_\varepsilon + \sigma(\theta_\varepsilon)(u_\varepsilon - \theta_\varepsilon) = 0 \\ \\ \varepsilon^2 \mathcal{E}_t(\theta_\varepsilon) = \sigma(\theta_\varepsilon)(\bar{u}_\varepsilon - \theta_\varepsilon), \qquad \bar{u}_\varepsilon = \dfrac{1}{4\pi} \displaystyle\int_{S^2} u_\varepsilon(t,x,\omega) d\omega. \end{cases}$$

Our results, already given in Theorems 2.2 and 2.3, will hence be proved under the following form:

Theorem 2.5. *Let us consider System* (4)*, posed for* $(t,x,\omega) \in (0,\tau) \times \Omega \times S^2$*, with initial conditions*

$$u_\varepsilon(0,x,\omega) = u^{\mathrm{in}}(x,\omega) \in L^\infty(\Omega \times S^2)$$
$$\theta_\varepsilon(0,x) = \theta^{\mathrm{in}}(x) \in L^\infty(\Omega)$$

and boundary data

$$u_\varepsilon(t,x,\omega)|_{\Gamma_-} = u^{\mathrm{b}}(t,x,\omega) \in L^\infty((0,\tau) \times \partial\Omega \times S^2),$$

where $\Omega \subset \mathbf{R}^3$ *is a bounded domain with boundary of class* C^1 *and* $\varepsilon > 0$*. Then System* (4) *admits a weak solution*

$$(u_\varepsilon(t,x,\omega), \theta_\varepsilon(t,x)) \in L^\infty((0,\tau) \times \Omega \times S^2) \times L^\infty((0,\tau) \times \Omega).$$

Theorem 2.6. *Let us consider System* (4)*, posed for* $(t,x,\omega) \in (0,\tau) \times \Omega \times S^2$*, with initial conditions*

$$u_\varepsilon(0,x,\omega) = u^{\mathrm{in}}(x,\omega) \in L^\infty(\Omega \times S^2)$$
$$\theta_\varepsilon(0,x) = \theta^{\mathrm{in}}(x) \in L^\infty(\Omega)$$

and boundary data

$$u_\varepsilon(t,x,\omega)|_{\Gamma_-} = u^{\mathrm{b}}(t,x) \in W^{1,\infty}((0,\tau) \times \partial\Omega),$$

where $\Omega \subset \mathbf{R}^3$ *is a bounded domain with boundary of class* C^1 *and* $\varepsilon > 0$*. Let* $\{u_\varepsilon\}$ *be a family of solutions of System* (4) *which satisfy the same initial and boundary conditions.*

Then, there exists a function $\rho \in L^\infty((0,\tau) \times \Omega)$ *such that* $\rho = \lim_{\varepsilon \to 0} \bar{u}_\varepsilon$ *strongly in* $L^2((0,\tau) \times \Omega)$*. The limit function* ρ *moreover solves the following initial-boundary value problem in* $(0,\tau) \times \Omega$*, for* $\tau > 0$*:*

$$\text{(5)} \qquad [\rho + \mathcal{E}(\rho)]_t - \nabla_x \cdot \left[\frac{\nabla_x \rho}{3\sigma(\rho)} \right] = 0.$$

with initial conditions $\rho(0,x) = \bar{u}^{\mathrm{in}}(x,\omega)$ *and boundary data* $\rho(t,x)|_{\partial\Omega} = u^{\mathrm{b}}(t,x)$*.*

3. Existence proof

This section is devoted to the proof of Theorem 2.5. In order to simplify the notation, in this section we will specialize system (4) by considering $\varepsilon = 1$, and by eliminating all the subscripts in the unknowns. Obviously, the results are valid for all parameters $\varepsilon > 0$.

3.1. **The maximum principle.** As a first step, we prove that (4) preserves the non-negativity of the solutions, which are uniformly bounded in time. This property is proved in the following proposition:

Proposition 3.1. *Let us suppose that there exists a solution (u, θ) for the radiative transfer system (4) in $(t, x, \omega) \in (0, \tau) \times \Omega \times S^2$, with bounded initial conditions*

$$u_\varepsilon(0, x, \omega) = u^{\mathrm{in}}(x, \omega) \in L^\infty(\Omega \times S^2)$$

$$\theta_\varepsilon(0, x) = \theta^{\mathrm{in}}(x) \in L^\infty(\Omega)$$

and bounded boundary data

$$u_\varepsilon(t, x, \omega)|_{\Gamma_-} = u^{\mathrm{b}}(t, x, \omega) \in L^\infty((0, \tau) \times \partial\Omega \times S^2).$$

Then there exists a constant M, which depends only on the initial and boundary data, such that $0 \le u, \theta \le M$ a.e. (uniformly in x, ω and t).

Proof. We develop first the part of the proof which establishes the bound from above. We define

$$M = \max\left\{ \|u^{\mathrm{in}}\|_{L^\infty(\Omega\times S^2)},\ \|\theta^{\mathrm{in}}\|_{L^\infty(\Omega)},\ \|u^{\mathrm{b}}\|_{L^\infty(\Gamma_-\times S^2\times(0,\tau])} \right\}.$$

We consider now the pair of constant functions $(v = M, S = M)$, solutions of the system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \omega \cdot \nabla_x v + \sigma(\theta)(v - S) = 0 \\[2ex] \dfrac{\partial}{\partial t}\mathcal{E}(S) = \sigma(\theta)(\bar{v} - S), \end{cases} \tag{6}$$

where θ satisfies the radiative transfer system (4), with initial and boundary conditions

$$v(x, \omega, t = 0) = M \qquad v(x, \omega, t)|_{\Gamma_-} = M.$$

$$S(x, t = 0) = M.$$

We subtract the first equation of system (6) from the first equation of system (4) and subtract the second equation of system (6) from the second equation of system (4).

We then multiply the two obtained equations by $\mathrm{sign}^+(u - v)$ and $\mathrm{sign}^+(\theta - S)$ respectively, where $\mathrm{sign}^+(y) = 1$ when $y \ge 0$ and $\mathrm{sign}^+(y) = 0$ when $y < 0$.

Finally we add the two equations and integrate with respect to x and ω in the domain $\Omega \times S^2$. We note that the term

$$\int_{\Omega\times S^2} \omega \cdot \nabla_x (u - v)\, d\omega dx = \int_{\Gamma\times S^2} (u - v)^+ \omega \cdot n_x \, ds d\omega,$$

where $(u - v)^+ = (u - v)\,\mathrm{sign}^+(u - v)$, is always non-negative: indeed, in the region $\Gamma_0 \cup \Gamma_+$, the factor $(\omega \cdot n_x)$ is non-negative, as well as $(u - v)^+$; on the other hand, in the region Γ_-, the term $(\omega \cdot n_x)$ is negative, but there $u < v$

because of the boundary conditions imposed to v, and therefore $(u-v)^+ = 0$ and the integral vanishes.

Since $\mathcal{E}'$ is non-negative, we can moreover use the identity

$$\operatorname{sign}^+(\theta - S) = \operatorname{sign}^+(\mathcal{E}(\theta) - \mathcal{E}(S)).$$

We hence obtain

$$\frac{d}{dt}\left[\int_{\Omega\times S^2}(u-v)^+\,d\omega dx + \int_{\Omega\times S^2}(\mathcal{E}(\theta)-\mathcal{E}(S))^+\,d\omega dx\right] \leq 0,$$

since

$$\int_{\Omega\times S^2}\sigma(\theta)(\theta-S)(\operatorname{sign}^+(u-v)-\operatorname{sign}^+(\theta-S))\,d\omega dx -$$
$$\int_{\Omega\times S^2}\sigma(\theta)(u-v)(\operatorname{sign}^+(u-v)-\operatorname{sign}^+(\theta-S))\,d\omega dx \leq 0.$$

But in $t=0$ we have that

$$\int_{\Omega\times S^2}(u(0)-v(0))^+\,d\omega dx + \int_{\Omega\times S^2}(\mathcal{E}(\theta(0))-\mathcal{E}(S(0)))^+\,d\omega dx = 0,$$

and therefore the bound from above is proved.

In order to prove that System (4) preserves the sign of the solution, we apply a similar strategy: we consider the function $\operatorname{sign}^-(y)$, defined as $\operatorname{sign}^-(y) = 0$ when $y \geq 0$ and $\operatorname{sign}^-(y) = -1$ when $y < 0$; then we multiply the first equation of (4) by $\operatorname{sign}^-(u)$ and the second one by $\operatorname{sign}^-(\theta)$. We integrate the equations in $\Omega \times S^2$ with respect to the space variable x and ω and deduce (notice that, by hypothesis on $\mathcal{E}$, the identity $\operatorname{sign}^-(\mathcal{E}(\theta)) = \operatorname{sign}^-(\theta)$ is always satisfied):

$$\frac{d}{dt}\int_{\Omega\times S^2}u^-\,dxd\omega + \int_{\Gamma\times S^2}u^-(\omega\cdot n_x)\,dsd\omega + \int_{\Omega\times S^2}\sigma(\theta)(u-\theta)\operatorname{sign}^-(u)\,dxd\omega = 0$$

and

$$\frac{d}{dt}\int_{\Omega\times S^2}(\mathcal{E}(\theta))^-\,dxd\omega = \int_{\Omega\times S^2}\sigma(\theta)(u-\theta)\operatorname{sign}^-(\theta)\,dxd\omega = 0,$$

where we have defined the function y^- as $y^- = 0$ when $y \geq 0$ and $y^- = -y$ when $y < 0$.

By summing the two equation, with the same strategy used to prove the maximum principle, we deduce that, separately,

$$\int_{\Gamma\times S^2}u^-(\omega\cdot n_x)\,dsd\omega \geq 0$$

and

$$\int_{\Omega\times S^2}\sigma(\theta)(u-\theta)(\operatorname{sign}^-(u)-\operatorname{sign}^-(\theta))\,dxd\omega \geq 0.$$

We can hence conclude that

$$\frac{d}{dt}\int_{\Omega\times S^2}[u^- + (\mathcal{E}(\theta))^-]\,dxd\omega \leq 0.$$

Since the initial and boundary conditions are non-negative, $\mathcal{E}$ is monotone and $\mathcal{E}(0) = 0$, this implies that $u \geq 0$ and $\mathcal{E}(\theta) \geq 0$ a.e. in Ω for all $t \in [0, \tau]$. □

3.2. **The truncated problem.** In the proof on the existence of a weak solution of System (4), we use a trick introduced by Mercier in [14], by working with the solutions of a truncated approximation of (4).

Then, thanks to the maximum principle of the truncated problem, it will be possible to construct sequences of solutions of such approximated problems which will be relatively compact in L^∞-weak*. Once passed to the limit, the existence proof will be achieved if one proves that the limit of the solutions of the approximated problems solves System (4).

We first consider the following truncated approximation of System (4):

$$
\begin{cases}
\dfrac{\partial u_n}{\partial t} + \omega \cdot \nabla_x u_n + \sigma_n(\theta_n)(U_n - \theta_n) = 0 \\[2ex]
\mathcal{E}_t(\theta_n) = \sigma_n(\theta_n)(\bar{U}_n - \theta_n), \qquad \bar{U}_n = \dfrac{1}{4\pi} \displaystyle\int_{S^2} U_n(t, x, \omega) d\omega,
\end{cases}
\tag{7}
$$

where

$$
\sigma_n(y) = \min\{\sigma(y), n\}, \qquad U_n(t, x, \omega) = \max\{0\,,\, \min\{u_n\,,\, M\}\}, \qquad n \in \mathbf{N}.
$$

System (7) has the form of a Lipschitzian perturbation of the free transport equations. Therefore, thanks to Theorems 6.1.2 and 6.1.6 in [17], System (7) admits a strong solution $(u_n, \theta_n) \in C^0([0, \tau]; L^2(\Omega \times S^2)) \times C^0([0, \tau]; L^2(\Omega))$.

By applying the same strategy of Lemma 3.1, we can prove that, if we suppose that there exists a solution (u_n, θ_n) for the truncated approximation (7) in $(t, x, \omega) \in (0, \tau) \times \Omega \times S^2$, with the same initial and boundary conditions of Theorem 2.5, then there exists a constant M, which depends only on the initial and boundary data, such that $0 \leq u_n, \theta_n \leq M$ a.e. in Ω, uniformly in x, ω and t.

Thanks to the maximum principle for System (7), the solution of such set of equations coincides therefore with the solution of the following system:

$$
\begin{cases}
\dfrac{\partial u_n}{\partial t} + \omega \cdot \nabla_x u_n + \sigma_n(\theta_n)(u_n - \theta_n) = 0 \\[2ex]
\mathcal{E}_t(\theta_n) = \sigma_n(\theta_n)(\bar{u}_n - \theta_n), \qquad \bar{u}_n = \dfrac{1}{4\pi} \displaystyle\int_{S^2} u_n(t, x, \omega) d\omega,
\end{cases}
\tag{8}
$$

where

$$
\sigma_n(y) = \min\{\sigma(y), n\}, \qquad n \in \mathbf{N}.
$$

We hence deduce the following Lemma:

Lemma 3.2. *Consider System* (8) *for* $(t, x, \omega) \in (0, \tau) \times \Omega \times S^2$, *with initial conditions*

$$
u_n(0, x, \omega) = u^{\text{in}}(x, \omega) \in L^\infty(\Omega \times S^2)
$$

$$\theta_n(0,x) = \theta^{\rm in}(x) \in L^\infty(\Omega)$$

and boundary data

$$u_n(t,x,\omega)|_{\Gamma_-} = u^{\rm b}(t,x,\omega) \in L^\infty((0,\tau)\times\partial\Omega\times S^2).$$

Then, there exists a strong solution $(u_n(t,x,\omega),\theta_n(t,x))$ *of System* (8) *which belongs to the space* $C^0([0,\tau];L^2(\Omega\times S^2))\times C^0([0,\tau];L^2(\Omega))$. *Moreover, there exists a constant* $M>0$ *such that*

$$\|u_n\|_{L^\infty((0,\tau)\times\Omega\times S^2)}, \|\theta_n\|_{L^\infty((0,\tau)\times\Omega\times S^2)} \le M$$

uniformly with respect to n.

Thanks to Lemma 3.2, the sequences $\{u_n\}$ and $\{\theta_n\}$ are relatively compacts in L^∞-weak*. Nevertheless, this property is not enough to pass to the limit because of the nonlinearities of the problem. We need strong compactness for the sequences $\{\bar u_n\}$ (obtained through an averaging lemma) and $\{\theta_n\}$.

We prove therefore the following result:

Lemma 3.3. *Let* $\{u_n\}$ *and* $\{\theta_n\}$ *be sequences of solutions of the truncated approximation* (8). *Then* $\bar u_n$ *is relatively compact in* $L^p_{\rm loc}(t,x)$ *for all* $p\in[1,+\infty)$.

Proof. Since θ_n is uniformly bounded in $L^\infty_{t,x}$ and the internal energy $\mathcal{E}$ is of class $C^0([0,+\infty))$, the second equation of System (8) shows that we deduce that

$$\int_\Omega \mathcal{E}(\theta_n)(\tau,x)\,dx - \int_\Omega \mathcal{E}(\theta_n)(0,x)\,dx + \\ \int_0^\tau\int_\Omega \sigma_n(\theta_n)\,\theta_n\,dtdx = \int_0^\tau\int_\Omega \sigma_n(\theta_n)\,\bar u_n\,dtdx.$$

Therefore, there exists a constant $C>0$ such that

$$\int_{(0,\tau)\times\Omega\times S^2} \sigma_n(\theta_n)u_n\,dtdxd\omega \le C.$$

By the first equation of System (8), we hence deduce that $(\partial_t+\omega\cdot\nabla_x)u_n$ is bounded in $L^1_{\rm loc}(t,x,\omega)$.

Since u_n bounded in $L^\infty(t,x,\omega)$, we then use simply the averaging lemma proved in [9] to conclude. □

As a next step we prove that the unknown θ_n of System (8) depends only on $\bar u_n$ in a very special way:

Lemma 3.4. *Consider System* (8). *Then there exists a ...functional* $\mathcal{F}$: $L^p_{\rm loc}((0,\tau)\times\Omega)\to L^p_{\rm loc}((0,\tau)\times\Omega)$, $p\in[1,+\infty)$ *such that* $\theta_n=\mathcal{F}(\bar u_n)$.

Proof. Let us consider the second equation of (8). Since $\mathcal{E}$ is of class C^1, we can introduce the function $H_n = H_n(\theta_n)$ defined in such a way that

$$H_n'(\theta_n) = \frac{\mathcal{E}'(\theta_n)}{\sigma_n(\theta_n)}. \tag{9}$$

This allows us to rewrite the second equation of (8) in the form

$$\frac{\partial H_n}{\partial t}(\theta_n) + \theta_n = \bar{u}_n. \tag{10}$$

Thanks to the hypotheses on $\mathcal{E}$ and σ_n, we can deduce that H_n is one-to-one and of class C^1. Indeed, by definition of H_n', we have that H_n' is always non-negative. The only point which can give troubles is $\theta_n = 0$. But

$$\lim_{\theta\to 0} H_n'(\theta_n) = \lim_{\theta_n\to 0} \frac{\mathcal{E}'(\theta_n)}{\sigma_n(\theta_n)} \sim \theta_n^{\alpha+\beta},$$

because $\mathcal{E}$ and σ are admissible internal energy and opacity respectively.

Since $\beta \in (0, 1/4]$ and $\alpha \in [-3/4, -\beta]$, we can deduce that

$$\lim_{y\to 0} (H_n^{-1})'(y)$$

exists and is always finite.

The quantity H_n being defined by integration of H_n', we can choose the arbitrary constant; the best choice is to put $H_n(0) = 0$:

$$H_n(\theta_n) = \int_0^{\theta_n} \frac{\mathcal{E}'(\zeta)}{\sigma_n(\zeta)}\, d\zeta.$$

Thanks to the maximum principle, the behavior of H_n when $\theta_n \to +\infty$ plays no role. We have then proved that $H_n \in C^1([0, H_n(M)])$, $H_n^{-1} \in C^1([0, M])$ and H_n is invertible. By setting

$$\phi_n = H_n(\theta_n),$$

Equation (10) becomes

$$\frac{\partial \phi_n}{\partial t} + H_n^{-1}(\phi_n) = \bar{u}_n. \tag{11}$$

We build the difference between the previous equation and a shifted version of the same equation. We can deduce that

$$\frac{1}{2}\frac{d}{dt}[\phi_n(t, x+h) - \phi_n(t, x)]^2 +$$
$$[H_n^{-1}(\phi_n(t, x+h)) - H_n^{-1}(\phi_n(t, x))][\phi_n(t, x+h) - \phi_n(t, x)] =$$
$$[\bar{u}_n(t, x+h) - \bar{u}_n(t, x)][\phi_n(t, x+h) - \phi_n(t, x)].$$

We integrate respect to x in Ω and obtain, thanks to the monotonicity of H_n (which is a consequence of the monotonicity of $\mathcal{E}$):

$$\frac{1}{2}\|\phi_n(t, x+h) - \phi_n(t, x)\|_{L^2_x}^2 \le \frac{1}{2}\|\phi_n(0, x+h) - \phi_n(0, x)\|_{L^2_x}^2 +$$
$$\int_0^t \int_\Omega [\bar{u}_n(s, x+h) - \bar{u}_n(s, x)][\phi_n(s, x+h) - \phi_n(s, x)]\, ds dx$$
$$\le \frac{1}{2}\|\phi_n(0, x+h) - \phi_n(0, x)\|_{L^2_x}^2 +$$
$$\int_0^t \|\bar{u}_n(s, \cdot + h) - \bar{u}_n(s, \cdot)\|_{L^2_x} \|\phi_n(s, \cdot + h) - \phi_n(\cdot, x)\|_{L^2_x}\, ds.$$

Thanks to Gronwall's Lemma, in the limit $|h| \to 0$ we obtain that

$$\|\phi_n(t, \cdot + h) - \phi_n(t, \cdot)\|_{L^2_x}^2 \to 0,$$

Moreover, since $\bar{u}_n$ is relatively compact in L^p_{loc}, $p \in [1, +\infty)$,

$$\int_0^t \|\bar{u}_n(s, \cdot + h) - \bar{u}_n(s, \cdot)\|_{L^2_x} ds \to 0$$

when $|h| \to 0$. Finally, since $\bar{u}_n$ is bounded in $L^\infty_{t,x}$ and $H_n^{-1}(\phi_n)$ is also bounded in $L^\infty_{t,x}$, we are guaranteed that $\dot{\phi}_n$ is bounded in $L^\infty_{t,x}$. Hence, we can deduce that the family of functions $\{\phi_n\}$ is relatively compact in $L^p_{\text{loc}}((0, \tau) \times \Omega)$, and therefore there exists a subsequence, which converges to a limit point ϕ in $L^p_{\text{loc}}((0, \tau) \times \Omega)$ and a.e.. □

3.3. **Proof of theorem 2.5.** Thanks to Lemma 3.2 and the Banach-Alaouglu theorem, u_n and θ_n are weakly* compact. This means that, by subsequences, there exists two limits u and θ such that

$$u_n \rightharpoonup u \in L^\infty([0, \tau] \times \Omega \times S^2) \qquad \theta_n \rightharpoonup \theta \in L^\infty([0, \tau] \times \Omega)$$

weakly-*.

The next step consists in proving that, if $\{u_n\}$ and $\{\theta_n\}$ are two sequences of solutions of System (8) with initial and boundary data as in Theorem 2.5, then the two weak* limits u and θ solve the non-truncated version of Equations (4) with initial and boundary conditions

$$u(x, \omega, t = 0) = u^{\text{in}}(x, \omega) \qquad u(x, \omega, t)|_{\Gamma_-} = u^{\text{b}}(t, x, \omega)$$

and

$$\theta(x, t = 0) = \theta^{\text{in}}(x).$$

The previous equations are highly nonlinear; hence the passage to the limit is not straightforward. We analyze therefore each term of the two equations.

i) We have easily that

$$\sigma_n(\theta_n)\, \theta_n = O(1)$$

in $L^\infty([0, \tau] \times \Omega)$: indeed, in any interval of type $[a, +\infty)$ with $a > 0$, σ is uniformly bounded and, when $y \to 0$, we have that

$$\sigma_n(y)\, y \sim y^{1-\beta}$$

with $\beta \in [0, 1/4)$.

ii) We look now at the term

$$\sigma_n(\theta_n)\, u_n.$$

Thanks to the second equation of system (8), we deduce, by integrating with respect to (t, x, ω) in $[0, \tau] \times \Omega \times S^2)$ that

$$\frac{1}{4\pi} \int_0^\tau \int_\Omega \int_{S^2} \sigma_n(\theta_n)\, u_n \, dt dx d\omega = \int_0^\tau \int_\Omega \sigma_n(\theta_n)\, \bar{u}_n \, dt dx =$$

$$\int_0^\tau \int_\Omega [\mathcal{E}_t(\theta_n) + \sigma_n(\theta_n)\,\theta_n]\,dtdx.$$

We already know that $\sigma_n(\theta_n)\,\theta_n$ belongs to $L^\infty([0,\tau]\times\Omega)$. We now consider the other term of the last integral: we have that

$$\int_0^\tau \int_\Omega \mathcal{E}_t(\theta_n)\,dtdx = \int_\Omega \mathcal{E}(\theta_n)(x,\tau)\,dx - \int_\Omega \mathcal{E}(\theta^{\text{in}})(x)\,dx.$$

Thanks to the hypotheses on $\mathcal{E}$, we have proved that also $\sigma_n(\theta_n)\,u_n$ belongs to $L^\infty([0,\tau]\times\Omega)$ since Ω is bounded.

In order to pass to the limit, we use the compactness properties of $\bar{u}_n$ and θ_n. By Lemma 3.3, we have that $\bar{u}_n$ is relatively compact in $L^p_{\text{loc}}((0,\tau)\times\Omega)$ for all $p\in[1,+\infty)$ and, by Lemma 3.4, that θ_n is relatively compact in $L^p_{\text{loc}}((0,\tau)\times\Omega)$ for all $p\in[1,+\infty)$.

We know that

(1) $u_n \rightharpoonup u$ in $L^\infty_{t,x,\omega}$-weak*
(2) $\theta_n \rightharpoonup \theta$ in $L^\infty_{t,x,\omega}$-weak* and, by Lemma 3.4, that $\theta_n \to \theta$ in L^p_{loc} and a.e..

Consequently, $\sigma_n(\theta_n)\,\theta_n \rightharpoonup \sigma(\theta)\,\theta$ in in L^p_{loc} and a.e..

We can now apply the dominated convergence theorem to control the nonlinear terms. We recall moreover that the only term which depends on the angular variable ω is u_n. Therefore we can deduce that, modulo extraction of a subsequence, there exists a measure μ such that

$$\sigma_n(\theta_n)\,u_n \rightharpoonup^* \mu.$$

We only need to identify μ with the limit product $\sigma(\theta)\,u$.

For any $m<n$, we have that

$$\sigma_m(\theta_n)\,u_n \le \sigma_n(\theta_n)\,u_n,$$

and therefore, in the limit $m,n\to+\infty$

$$\sigma(\theta)\,u \le \mu.$$

On the other hand, the second equation of (8), says us that

$$\int_\Omega \mathcal{E}(\theta_n)(\tau,x)\,dx - \int_\Omega \mathcal{E}(\theta^{\text{in}})(x)\,dx + \int_0^\tau\int_\Omega \sigma_n(\theta_n)\,\theta_n\,dtdx =$$
$$\frac{1}{4\pi}\int_0^\tau\int_\Omega\int_{S^2} \sigma_n(\theta_n)\,u_n\,dtdxd\omega.$$

The l.h.s converges to

$$\int_\Omega \mathcal{E}(\theta)(\tau,x)\,dx - \int_\Omega \mathcal{E}(\theta^{\text{in}})(x)\,dx + \int_0^\tau\int_\Omega \sigma(\theta)\,\theta\,dtdx,$$

whereas the r.h.s converges to

$$\int_0^\tau\int_\Omega \mu\,dtdx.$$

But in the limit, by strong convergence

$$\sigma(\theta)\,\bar{u} = \mathcal{E}_t(\theta) + \sigma(\theta)\,\theta,$$

and therefore

$$\int_0^\tau \int_\Omega \sigma(\theta)\theta\, u\, dtdx = \int_0^\tau \int_\Omega \mu\, dtdx.$$

We have hence proved that the term $\sigma_n(\theta_n)\,u_n$ converges weakly* to $\sigma(\theta)\,u$.

Hence, the limit (u,θ) of the sequence (u_n,θ_n) solves System (4).

iii) The proof will be complete once we show that the initial and boundary conditions for the limit problem are satisfied (in the sense of traces).

We know that the sequence u_n is bounded in $L^p((0,\tau)\times\Omega\times S^2)$ for any $p\in[1,+\infty)$. If we can find an exponent $q>1$ such that $(\partial_t+\omega\cdot\nabla_x)u_n$ is bounded in $L^q((0,\tau)\times\Omega\times S^2)$, we can deduce that, thanks to Cessenat's results [4, 5], u_n has a trace in L^q-sense and that $u_n \rightharpoonup u$ weakly* in $L^q(\{t=0\}\times\Omega\times S^2\cup\partial\Omega\times S^2\times(0,\tau])$.

Let us therefore consider the first equation of System (8). Thanks to the behavior in zero of σ_n, we deduce that $\sigma_n(\theta_n)\,\theta_n$ is bounded in $L^\infty((0,\tau)\times\Omega)$. Therefore we deduce that $(\partial_t+\omega\cdot\nabla_x)u_n$ is bounded in $L^q((0,\tau)\times\Omega\times S^2)$ if we can prove that $\sigma_n(\theta_n)\,u_n\in L^q((0,\tau)\times\Omega\times S^2)$; we have

$$\int_{(0,\tau)\times\Omega\times S^2} \sigma_n^q(\theta_n)\,u_n^q\,dtdxd\omega \le \|u_n\|_{L^\infty}^{q-1}\int_{(0,\tau)\times\Omega}\sigma_n^q(\theta_n)\,\bar{u}_n\,dtdx.$$

By the second equation of System (8), we have

$$\int_{(0,\tau)\times\Omega}\sigma_n^{q-1}(\theta_n)\mathcal{E}'(\theta_n)\frac{\partial\theta_n}{\partial t}\,dtdx + \int_{(0,\tau)\times\Omega}\sigma_n^q(\theta_n)\,\theta\,dtdx =$$

$$\int_{(0,\tau)\times\Omega}\sigma_n^q(\theta_n)\,\bar{u}_n\,dtdx.$$

But the first term in the l.h.s. of the previous equation is bounded thanks to fundamental theorem of integral calculus and the second term in the l.h.s. is bounded thanks to the behavior in zero of σ_n, provided that $q\in(1,4]$. Hence, we deduce that

$$\int_{(0,\tau)\times\Omega\times S^2}\sigma_n^q(\theta_n)\,u_n^q\,dtdxd\omega$$

is bounded and therefore $u_n \rightharpoonup u$ weakly* in $L^q(\{t=0\}\times\Omega\times S^2\cup\partial\Omega\times S^2\times(0,\tau])$ for any $q\in(1,4]$.

In order to prove, finally, that the limit θ satisfies the initial condition, we consider the second equation of System (4). Since $\mathcal{E}$ is of class C^1 and strictly increasing and the second member of the equation is bounded in L^p_{loc} and a.e., we deduce that $\theta(x,t)\to\theta^{\text{in}}(x)$ strongly in $L^p(\Omega)$ for all $1\le p<\infty$.

Hence the proof of Theorem 2.5 is complete.

4. Relative entropy estimates

This section is devoted to give the tools which will permit to handle the limiting behavior of system (4).

From now on, we set

$$\nu = \|u^{\mathrm{b}}\|_{W^{1,\infty}(\partial\Omega\times[0,\tau])} \text{ and } m = \min_{0\le y\le M} \sigma(y). \tag{12}$$

The main mathematical tool we will need is a functional, which is nothing but the relative entropy with respect to a gauge profile.

There are many ways to define this gauge profile. Its most important property concerns the behavior on the boundary. For example, the gauge function $f = f(t,x)$ could be the solution of the Laplace equation with boundary data given by $u^{\mathrm{b}}(t,x)$:

$$\begin{cases} \Delta_x f(t,x) = 0 \qquad x \in \Omega \\ \\ f(t,x)|_{\partial\Omega} = u^{\mathrm{b}}(t,x) \in L^\infty((0,\tau]\times\partial\Omega). \end{cases} \tag{13}$$

Classical theory on the Laplace equation ensures that f exists, is unique and satisfies the strong maximum principle. In particular, $0 \le f \le M = \|u^{\mathrm{b}}\|_\infty$ in $\bar\Omega$ and $f \in W^{2,p}(\Omega)$ for any $p \in [1,+\infty)$. Obviously, here the time t acts only as a parameter.

4.1. An elementary inequality.

The natural entropy generating function to control the L^2-norm of radiation flux should be $y^2/2$.

However, one needs to control boundary terms, and the natural choice stated before does not seems the best one. To do so, it is more convenient to use the following entropy generating function

$$\Phi(y) = \frac{1}{2}y^2 + \frac{1}{2}(\nu+1)^2\,,$$

where ν has been introduced in (12). With this definition $\Phi(y) \ge (\nu+1)y = \Phi'(\nu)y + y$, i.e.

$$\Phi(y) - \Phi'(\nu)y \ge y \tag{14}$$

and this last inequality is helpful in handling boundary terms.

4.2. Entropy inequalities.

In this subsection we show in which way the control on radiation flux come from a entropy production control.

We multiply the first equation of system (4) by $\Phi'(u_\varepsilon)$ and integrate with respect to x and ω in $\Omega\times S^2$. We obtain:

$$\varepsilon^2 \frac{d}{dt}\int_{\Omega\times S^2} \Phi(u_\varepsilon)\,d\omega\,dx + \varepsilon\int_{\partial\Omega\times S^2} \omega\cdot n_x \Phi(u_\varepsilon)\,d\omega\,ds(x) +$$

$$\int_{\Omega\times S^2} \sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)\Phi'(u_\varepsilon)\,d\omega\,dx = 0. \tag{15}$$

We write now Equation (15) in the form

$$\frac{d}{dt}\int_{\Omega\times S^2}\Phi(u_\varepsilon)\,d\omega\,dx+\frac{1}{\varepsilon}\int_{\partial\Omega\times S^2}\omega\cdot n_x[\Phi(u_\varepsilon)-\Phi'(f)u_\varepsilon]\,d\omega\,ds(x)=$$

$$-\frac{1}{\varepsilon^2}\int_{\Omega\times S^2}\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)\Phi'(u_\varepsilon)\,d\omega\,dx-\frac{1}{\varepsilon}\int_{\Omega\times S^2}\nabla\cdot(\omega\Phi'(f)u_\varepsilon)\,d\omega\,dx. \tag{16}$$

Let us study now the various terms of the previous equation.

(i) We have that

$$\int_{\partial\Omega\times S^2}\omega\cdot n_x[\Phi(u_\varepsilon)-\Phi'(f)u_\varepsilon]\,d\omega\,ds=\int_{\partial\Omega}\int_{\omega\cdot n_x>0}\omega\cdot n_x[\Phi(u_\varepsilon)-\Phi'(f)u_\varepsilon]\,d\omega\,ds$$

$$+\int_{\partial\Omega}\int_{\omega\cdot n_x<0}\omega\cdot n_x[\Phi(u_\varepsilon)-\Phi'(f)u_\varepsilon]\,d\omega\,ds\geq$$

$$\int_{\partial\Omega}\int_{\omega\cdot n_x>0}\omega\cdot n_x[\Phi(f)-\Phi'(f)f]\,d\omega\,ds+\int_{\partial\Omega}\int_{\omega\cdot n_x<0}\omega\cdot n_x[\Phi(u_\varepsilon)-\Phi'(f)u_\varepsilon]\,d\omega\,ds$$

$$=\int_{\partial\Omega}(\Phi(f)-\Phi'(f)f)\left[\int_{S^2}\omega\cdot n_x\,d\omega\right]\,ds=0.$$

Here we have used the convexity of Φ, inequality (14) and the fact that $[\Phi(f)-\Phi'(f)f]$ does not depend on ω.

This term is therefore non-negative.

(ii) We perform now some manipulations of the last term in Equation (16):

$$\frac{1}{\varepsilon}\int_{\Omega\times S^2}\nabla\cdot(\omega\Phi'(f)u_\varepsilon)\,d\omega\,dx=\frac{1}{\varepsilon}\int_{\Omega\times S^2}\omega\cdot[u_\varepsilon\Phi''(f)\nabla_x f+\Phi'(f)\nabla_x u_\varepsilon]\,d\omega\,dx=$$

$$\frac{1}{\varepsilon}\int_{\Omega\times S^2}[u_\varepsilon\Phi''(f)\omega\cdot\nabla_x f]\,d\omega\,dx+\int_\Omega\Phi'(f)(\nabla_x\cdot j_\varepsilon)\,dx.$$

We notice that

$$\int_\Omega\Phi'(f)(\nabla_x\cdot j_\varepsilon)\,dx=\int_\Omega\Phi'(f)\left[\frac{\partial}{\partial t}\mathcal{E}(\theta_\varepsilon)-\frac{\partial}{\partial t}\rho_\varepsilon\right]\,dx=$$

$$\int_\Omega\left[\Phi'(f)\mathcal{E}(\theta_\varepsilon)\right]_t\,dx-\int_\Omega\Phi''(f)\mathcal{E}(\theta_\varepsilon)f_t\,dx-$$

$$\int_\Omega\left[\Phi'(f)\rho_\varepsilon\right]_t\,dx+\int_\Omega\Phi''(f)\rho_\varepsilon f_t\,dx.$$

Also the other term can be controlled: we have that

$$\frac{1}{\varepsilon}\int_{\Omega\times S^2}[u_\varepsilon\Phi''(f)\omega\cdot\nabla_x f]\,d\omega\,dx=\int_{\Omega\times S^2}[\Phi''(f)j_\varepsilon\cdot\nabla_x f]\,d\omega\,dx\leq$$

$$\frac{1}{2\gamma}\int_\Omega|\nabla_x\Phi'(f)|^2\,dx+\gamma\int_\Omega|j_\varepsilon|^2\,dx.$$

(iii) Finally, we deduce that

$$\frac{1}{\varepsilon^2}\int_{\Omega\times S^2}\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)\Phi'(u_\varepsilon)\,d\omega\,dx=$$

$$\frac{1}{\varepsilon^2}\int_{\Omega\times S^2}\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)[\Phi'(u_\varepsilon)-\Phi'(\theta_\varepsilon)]\,d\omega\,dx+\int_{\Omega\times S^2}\mathcal{E}_t(\theta_\varepsilon)\Phi'(\theta_\varepsilon)\,dx.$$

If we remember the form of Φ, we see immediately that the first member in the right-hand side of the previous equation controls the L^2-norm of the flux:

$$\frac{1}{\varepsilon^2}\int_{\Omega\times S^2}\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)^2\,d\omega\,dx\geq$$

$$\frac{1}{\varepsilon^2}\int_{\Omega}\sigma(\theta_\varepsilon)\left[\int_{S^2}(u_\varepsilon-\theta_\varepsilon)^2\,d\omega\int_{S^2}(\omega^{(i)})^2\,d\omega\right]\,dx\geq$$

$$\frac{1}{\varepsilon^2}\int_{\Omega}\sigma(\theta_\varepsilon)\left[\int_{S^2}(u_\varepsilon-\theta_\varepsilon)\omega^{(i)}\,d\omega\right]^2\,dx=\frac{1}{\varepsilon^2}\int_{\Omega}\sigma(\theta_\varepsilon)\left[\int_{S^2}u_\varepsilon\omega^{(i)}\,d\omega\right]^2\,dx=$$

$$\int_{\Omega}\sigma(\theta_\varepsilon)(j_\varepsilon^{(i)})^2\,dx.$$

Hence

$$\frac{1}{\varepsilon^2}\int_{\Omega\times S^2}\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)^2\,d\omega\,dx\geq\frac{1}{3}\int_{\Omega}\sigma(\theta_\varepsilon)|j_\varepsilon|^2\,dx\geq\frac{m}{3}\int_{\Omega}|j_\varepsilon|^2\,dx.$$

We therefore deduce that Equation (16) implies

$$\frac{d}{dt}\int_{\Omega\times S^2}\left[\frac{1}{2}u_\varepsilon^2+f(\mathcal{E}(\theta_\varepsilon)-\rho_\varepsilon)\right]\,dxd\omega\leq\left(\gamma-\frac{m}{3}\right)\int_{\Omega}|j_\varepsilon|^2\,dx-$$

$$\int_{\Omega}\mathcal{E}_t(\theta_\varepsilon)\theta_\varepsilon\,dx-\int_{\Omega}f_t\left[\rho_\varepsilon-\mathcal{E}(\theta_\varepsilon)\right]\,dx+\frac{1}{2\gamma}\int_{\Omega}|\nabla_x f|^2\,dx.$$

If we choose $\gamma<m/3$ in the previous inequality and denote by

$$Q(y)=\int_0^y\mathcal{E}(s)s\,ds,$$

we deduce that

$$\frac{d}{dt}\int_{\Omega\times S^2}\left[\frac{1}{2}u_\varepsilon^2+f(\mathcal{E}(\theta_\varepsilon)-\rho_\varepsilon)+Q(\theta_\varepsilon)\right]\,dxd\omega+\left(\frac{m}{3}-\gamma\right)\int_{\Omega}|j_\varepsilon|^2\,dx\leq C,$$

where C is a positive constant which can be computed explicitly. We have then deduced the following result:

Proposition 4.1. *Let $(u_\varepsilon,\theta_\varepsilon)$ be a solution of the scaled system (4), in $(0,\tau)\times\Omega\times S^2$, with initial conditions*

$$u_\varepsilon(0,x,\omega)=u^{\mathrm{in}}(x,\omega)\in L^\infty(\Omega\times S^2)$$
$$\theta_\varepsilon(0,x)=\theta^{\mathrm{in}}(x)\in L^\infty(\Omega)$$

and boundary data

$$u_\varepsilon(t,x,\omega)|_{\Gamma_-}=u^{\mathrm{b}}(t,x,\omega)\in W^{1,\infty}((0,\tau)\times\partial\Omega\times S^2).$$

Then there exists a positive constant $J=J(\tau,u^{\mathrm{b}},u^{\mathrm{in}},\theta^{\mathrm{in}},m)$ such that the current j_ε satisfies

$$\int_0^\tau\int_\Omega|j_\varepsilon(t,x)|^2\,dt\,dx\leq J$$

and

$$\frac{1}{\varepsilon^2}\int_0^\tau \int_{\Omega\times S^2} \sigma(\theta_\varepsilon)(u_\varepsilon - \theta_\varepsilon)^2 \, dt \, dx \, d\omega \leq J$$

uniformly for each $\varepsilon > 0$.

Thanks to Proposition 4.1, we can deduce the following corollary:

Corollary 4.2. *Let* $(u_\varepsilon, \theta_\varepsilon)$ *as in Proposition* 4.1*. Then, if one of the sequences* u_ϵ*, or* θ_ϵ*, or* $\bar{u}_\epsilon$ *admits a limit, then all the three sequences go to the same limit in strong* L^2*-sense.*

Proof. Since σ is bounded below by a strictly positive constant, we obtain from Proposition 4.1 that

$$\frac{1}{\varepsilon}\|u_\varepsilon - \theta_\varepsilon\|_{L^2_{t,x,\omega}} = O(1)$$

and therefore

$$\lim_{\epsilon\to 0}\|u_\epsilon - \theta_\epsilon\|_{L^2_{t,x,\omega}} = 0.$$

Since

$$\|\bar{u}_\varepsilon - \theta_\varepsilon\|_{L^2_{t,x}} \leq \|u_\varepsilon - \theta_\varepsilon\|_{L^2_{t,x,\omega}}$$

and

$$\|u_\varepsilon - \bar{u}_\varepsilon\|_{L^2_{t,x,\omega}} \leq \|u_\varepsilon - \theta_\varepsilon\|_{L^2_{t,x,\omega}} + \|\bar{u}_\varepsilon - \theta_\varepsilon\|_{L^2_{t,x}},$$

we obtain that also

$$\lim_{\epsilon\to 0}\|u_\epsilon - \theta_\epsilon\|_{L^2_{t,x,\omega}} = \lim_{\epsilon\to 0}\|\theta_\epsilon - \bar{u}_\epsilon\|_{L^2_{t,x}} = 0.$$

□

5. Convergence to the Rosseland equation

This section is devoted to conclude the proof on the convergence of the diffusive limit stated in Theorem 2.6. Since we will use various relatively compact sequences, when we will indicate that a sequence converges to a limit, we will mean that there exists a subsequences which converges to the limit.

Let $(u^{\mathrm{in}}, \theta^{\mathrm{in}})$ and u^{b} be initial and boundary data which satisfy the hypotheses of Theorem 2.6; then, for each $\epsilon > 0$, let $(u_\epsilon, \theta_\epsilon)$ be the solution to the scaled system (4).

It follows from Propositions 3.1 and 4.1 that, for each $\tau > 0$, one has

$$\|\rho_\epsilon\|_{L^\infty((0,\tau)\times\Omega)} \leq K$$

and

$$\|j_\epsilon\|_{L^2([0,\tau]\times\Omega)} \leq J^{1/2}$$

for each $\epsilon > 0$.

By the Banach-Alaoglu theorem, for each $\tau > 0$

(17) the family ρ_ϵ is relatively compact in $L^\infty((0,T)\times\Omega)$ weak-*,

and therefore there exists $\rho \in L^\infty((0,\tau)\times\Omega)$ such that $\rho_\epsilon \rightharpoonup^* \rho$, while

$$\text{the family } j_\epsilon \text{ is relatively compact in } L^2([0,\tau]\times\Omega) \text{ weak,} \tag{18}$$

which means that there exists $j \in L^2((0,\tau)\times\Omega)$ such that $j_\epsilon \rightharpoonup j$.

Because of the nonlinearities, that are present both in the scaled system (4) and in the limiting nonlinear diffusion equation, weak compactness results as above are not enough to pass to the limit as $\epsilon \to 0$. Strong L^2 compactness of the family ρ_ϵ is required, which will be obtained by using Minty's trick [13] and compensated compactness.

Consider the vector fields

$$p_\epsilon = (\rho_\epsilon + \mathcal{E}(\theta_\epsilon), j_\epsilon) \text{ and } q_\epsilon = (\rho_\epsilon, 0, 0, 0)\,.$$

By the arguments as above, both vector fields satisfy

$$p_\epsilon \text{ and } q_\epsilon \text{ are bounded in } (L^2([0,\tau]\times\Omega))^4\,.$$

Moreover, if we integrate the first equation of system (4) with respect to ω in S^2, we obtain

$$\frac{\partial}{\partial t}\left[\frac{\rho_\varepsilon}{c} + \mathcal{E}(\theta_\varepsilon)\right] + \nabla_x \cdot j_\varepsilon = 0, \tag{19}$$

which means:

$$\operatorname{div}_{t,x} p_\epsilon = 0.$$

In order to prove that

$$\operatorname{curl}_{t,x} q_\epsilon \text{ is relatively compact in } H^{-1}_{t,x},$$

we simply use the averaging lemma [9], which give us compactness in x, the behavior of the time derivative being irrelevant, since the only non-zero component of q_ε is the first one.

Let $\mathcal{E}_*$ the weak-* limit of $\mathcal{E}(\theta_\varepsilon)$. Since $(\bar{u}_\varepsilon - \theta_\varepsilon)$ tends strongly to zero in L^2, we can deduce that also $\mathcal{E}(\rho_\varepsilon)$ converges weakly-* to $\mathcal{E}_*$. By compensated compactness (the div-curl lemma in [16]) we find that, by subsequences,

$$p_{\epsilon_n}\cdot q_{\epsilon_n} = \rho_{\epsilon_n}^2 + \rho_{\epsilon_n}\mathcal{E}(\theta_{\epsilon_n}) \rightharpoonup p\cdot q = \rho^2 + \rho\mathcal{E}_* \tag{20}$$

in the sense of Radon measures on $(0,T)\times(0,1)$ as $\epsilon_n \to 0$.

The family

$$\rho_\epsilon \text{ is relatively compact in } L^2([0,T]\times[0,1]) \text{ strong:} \tag{21}$$

indeed,

$$(\rho_\varepsilon - \rho)^2 + (\rho_\varepsilon - \rho)(\mathcal{E}(\rho_\varepsilon) - \mathcal{E}(\rho)) \geq 0$$

since $\mathcal{E}$ is monotone non-decreasing. Moreover, we deduce that, by (20),

$$(\rho_\varepsilon - \rho)^2 + (\rho_\varepsilon - \rho)(\mathcal{E}(\rho_\varepsilon) - \mathcal{E}(\rho)) \to 0$$

in $\mathcal{D}'$. A non-negative sequence which converges in $\mathcal{D}'$ to zero converges also strongly to zero. Since also the term

$$(\rho_\varepsilon - \rho)(\mathcal{E}(\rho_\varepsilon) - \mathcal{E}(\rho))$$

is non-negative, we deduce finally that $\rho_\varepsilon \to \rho$ (strongly) in $L^2((0,\tau)\times\Omega)$.

Because of the regularity on $\mathcal{E}$ (see Definition 2.4) and because $(\theta_\epsilon - \bar{u}_\epsilon) \to 0$ strongly in $L^2((0,\tau)\times\Omega)$, then $\mathcal{E}(\theta_\varepsilon) \to \mathcal{E}(\rho)$ (strongly) in $L^2((0,\tau)\times\Omega)$.

Let us now take into account the flux equation

$$\varepsilon^2 \frac{\partial j_\varepsilon^{(i)}}{\partial t} + \frac{1}{4\pi}\int_{S^2} \omega^{(i)} \nabla_x \cdot (\omega u_\varepsilon)\, d\omega + \sigma(\theta_\varepsilon)\, j_\varepsilon^{(i)} = 0. \tag{22}$$

Since σ is strictly positive, we can state that

$$\sigma(\theta_\varepsilon)^{1/2}\, j_\varepsilon = -\sigma(\theta_\varepsilon)^{-1/2}\left[\varepsilon^2 \frac{\partial j_\varepsilon}{\partial t} + \frac{1}{4\pi}\nabla_x \cdot \int_{S^2} (\omega\otimes\omega) u_\varepsilon\, d\omega\right].$$

Now, by Proposition (4.1), we have that

$$\|\sigma(\theta_\varepsilon)^{1/2} j_\varepsilon\|^2_{L^2((0,\tau)\times\Omega)} \le \|\sigma(\theta_\varepsilon)^{1/2}(u_\varepsilon - \theta_\varepsilon)\|^2_{L^2((0,\tau)\times\Omega\times S^2)} + \|\sigma(\theta_\varepsilon)^{1/2}\theta_\varepsilon\|^2_{L^2((0,\tau)\times\Omega)} \le \varepsilon^2 J + C\tau,$$

where C is the maximum of the function $\sigma(y)y^2$ for $y \in [0, M]$.

Therefore, there exists $\eta \in L^2((0,\tau)\times\Omega)$ such that $\sigma(\theta_\varepsilon)^{1/2} j_\varepsilon \rightharpoonup \eta$ in $L^2((0,\tau)\times\Omega)$. Since σ is strictly positive and $\theta_\varepsilon \to \rho$ strongly in $L^2((0,\tau)\times\Omega)$, we deduce that

$$j_\varepsilon \rightharpoonup \sigma(\theta_\varepsilon)^{-1/2}\eta = j.$$

On the other hand, if we multiply the first equation of the system (4) by u_ε, we deduce:

$$\frac{1}{2}\left(\varepsilon \frac{\partial u_\varepsilon^2}{\partial t} + \omega\cdot\nabla_x u_\varepsilon^2\right) = [u_\varepsilon \sigma(\theta_\varepsilon)^{1/2}]\left[\frac{1}{\epsilon}\sigma(\theta_\varepsilon)^{1/2}(u_\varepsilon - \theta_\varepsilon)\right]. \tag{23}$$

Thanks to Proposition 4.1, there exists a function $\zeta \in L^2((0,\tau)\times\Omega)$ such that

$$\frac{1}{\epsilon}\sigma(\theta_\varepsilon)^{1/2}(u_\varepsilon - \theta_\varepsilon) \rightharpoonup \zeta$$

in $L^2((0,\tau)\times\Omega)$ and therefore we can pass to the limit and obtain that

$$\zeta = \frac{1}{\sqrt{\sigma(\rho)}}\,\omega\cdot\nabla u,$$

since σ is bounded below by a positive constant by hypothesis.

But it is easy to see that

$$\eta = -\int_{S^2} \omega\,\zeta, \partial\omega = -\frac{1}{3\sqrt{\sigma(\rho)}}\,\nabla\rho,$$

and finally that

$$(\rho + \mathcal{E}(\rho))_t - \nabla_x \cdot \left[\frac{\nabla_x \rho}{3\sigma(\rho)}\right] = 0,$$

which is equation (5).

In order to verify that the initial and boundary conditions are the correct limits of the conditions imposed on the radiative transfer system, we will use a regularity argument.

Let us now consider the continuity equation (19): since the flux j_ε is bounded in $L^2((0,\tau)\times\Omega)$, we deduce that

$$(\rho_\varepsilon+\mathcal{E}(\theta_\varepsilon))_t \text{ is bounded in } L^2(0,T;H^{-1}(\Omega))\,;$$

which, joint to the bound on ρ_ϵ and $\mathcal{E}$, implies that

$$\rho_\epsilon+\mathcal{E}(\theta_\epsilon) \text{ is relatively compact in } C([0,\tau];H^{-1}(\Omega)) \tag{24}$$

by Arzela-Ascoli's theorem and therefore the target equation recovers the correct initial data.

Because of the initial condition of the radiative transfer system (4), one has

$$\rho\in C([0,\tau];H^{-1}(\Omega)) \text{ and } \rho\big|_{t=0}=\frac{1}{4\pi}\int_{S^2}u^{in}(x,\omega)\,d\omega. \tag{25}$$

Let us now care about the boundary conditions: we consider equation (23) and perform the limit as $\epsilon\to 0$. We note immediately, thanks to the identity

$$\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)u_\varepsilon=\sigma(\theta_\varepsilon)(u_\varepsilon-\theta_\varepsilon)^2+\sigma(\theta_\varepsilon)u_\varepsilon\theta_\varepsilon-\sigma(\theta_\varepsilon)\theta_\varepsilon^2,$$

that $\omega\cdot\nabla_x u_\varepsilon$ is bounded in $L^2((0,\tau)\times\Omega\times S^2)$, we can deduce that, thanks to Cessenat's results [4, 5], u_ε has a trace in L^2-sense and that $u_\varepsilon\rightharpoonup u$ weakly in $L^2((0,\tau]\times\partial\Omega\times S^2)$, and therefore the trace of the limit exists. Thus the proof of the nonlinear diffusion limit is competed.

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